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PARTICLE SYSTEMS WITH REPULSION EXPONENT β AND RANDOM MATRICES

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Abstract. We consider a class of particle systems generalizing the β-Ensembles from random matrix theory. In these new ensembles, particles experience repulsion of power $\beta > 0$ when getting close, which is the same as in the β-Ensembles. For distances larger than zero, the interaction is allowed to differ from those present for random eigenvalues. We show that the local bulk correlations of the β-Ensembles, universal in random matrix theory, also appear in these new ensembles. This result extends the bulk universality classes of random matrix theory and may lead to a better understanding of the occurrences of random matrix bulk statistics in several observations which have no obvious connection to random matrices. The present work is a generalization of [GV12] where a similar result was proved for $\beta = 2$.

1. Introduction and Main Results

Random matrix theory is well-known for universality phenomena which means that many essentially different matrix distributions lead in the limit of growing dimension to the same spectral statistics.

In the past 15 years or so, much progress has been made in proving universality of local spectral distributions, especially correlations between neighboring eigenvalues in the bulk of the spectrum and of the largest eigenvalues. It is known that there is a parameter, usually denoted β, which determines the universality class of the ensemble. To explain this in more detail, define for any $\beta > 0$ and a continuous function $Q : \mathbb{R} \longrightarrow \mathbb{R}$ of sufficient growth at infinity, the invariant β-Ensemble $P_{N,Q,\beta}$ on $\mathbb{R}^N$ which is given by

$$P_{N,Q,\beta}(x) := \frac{1}{Z_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta \, e^{-N \sum_{j=1}^N Q(x_j)}. \tag{1}$$

(With a slight abuse of notation, we will not distinguish between a measure and its density.) For $\beta = 1, 2, 4$, $P_{N,Q,\beta}$ is the eigenvalue distribution of a probability ensemble on the space of $(N \times N)$ matrices with real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$) or quaternionic self-dual ($\beta = 4$) entries, respectively. The matrix distributions are invariant under orthogonal, unitary or symplectic conjugations, respectively, explaining the name "invariant

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ensembles". For arbitrary β, only for quadratic Q, $P_{N,Q,\beta}$ is known to be an eigenvalue distribution.

It has been shown (see [GV12] for references) that the local spectral statistics in the bulk or at the edges of the spectrum do in many cases not depend on Q or, in other terms, invariant ensembles with different potentials Q but the same β have the same local statistics. It is also known that different values of β lead to different limiting (local) distributions. This is not surprising as the interaction term $\prod_{i<j} |x_i - x_j|^\beta$ has a strong effect on neighboring eigenvalues whereas $e^{-N\sum_{j=1}^N Q(x_j)}$ just confines all eigenvalues independently into a compact interval. In the limit $N \to \infty$, these two competing forces balance and produce a limiting measure of compact support.

In [GV12] the question was addressed whether the interaction term $\prod_{i<j} |x_i - x_j|^\beta$ could be changed without changing the local statistics. To this end, we introduced ensembles with density proportional to

$$\prod_{i<j} \varphi(x_i - x_j) e^{-N\sum_{j=1}^N Q(x_j)}, \tag{2}$$

where Q is a continuous function of sufficient growth at infinity compared to the continuous function $\varphi : \mathbb{R} \longrightarrow [0, \infty)$. The interaction potential φ fulfills

$$\varphi(0) = 0, \quad \varphi(t) > 0 \text{ for } t \neq 0 \quad \text{and} \quad \lim_{t\to 0} \frac{\varphi(t)}{|t|^\beta} = c > 0 \quad \text{for some } \beta > 0, \tag{3}$$

or, in other terms, 0 is the only zero of φ and it is of order β. It has been conjectured in [GV12] that the bulk correlations for the ensembles (2) are the same as in the case $\varphi(t) = |t|^\beta$, i.e. the same as for the invariant ensembles in random matrix theory. This was proved in [GV12] for $\beta = 2$ and a special class of functions φ and Q. In the present work, we prove a similar result for arbitrary $\beta > 0$. This shows that the local bulk correlations (at least in the considered cases) merely depend on the *repulsion exponent* β and not on the interaction of eigenvalues at distances larger than 0.

We believe that these results may lead to an explanation for the occurrence of random matrix bulk statistics in a number of seemingly unrelated observations in real world and science (see [GV12] for references). Spacings between cars in different situations were found to be fitted well by the universal spacing statistics from random matrix theory ($\beta = 1$ for parking along one-way streets, $\beta = 2$ along two-way streets, $\beta = 4$ for waiting in front of traffic signals). Also spacings between perching birds and between bus arrival times at stops in certain cities seem to obey ($\beta = 2$) random matrix spacing statistics. Gaps between the Riemann zeta function on the critical line are another famous example from mathematics (also $\beta = 2$). In all these observations, a natural repulsion between consecutive quantities is present.

Furthermore, the ensemble (2) does not seem to have a spectral interpretation which makes our findings a first step in proving universality of random matrix bulk distributions for more general particle systems.

To state our main results, we first rewrite the ensemble (2). Let h be a continuous even function which is bounded below. Let Q be a continuous even function of sufficient growth

at infinity. By $P^h_{N,Q,\beta}$ we will denote the probability density on $\mathbb{R}^N$ defined by

$$P^h_{N,Q,\beta}(x) := \frac{1}{Z^h_{N,Q,\beta}} \prod_{i<j} |x_i - x_j|^\beta \exp\{-N \sum_{j=1}^N Q(x_j) - \sum_{i<j} h(x_i - x_j)\}, \tag{4}$$

where $Z^h_{N,Q,\beta}$ denotes the normalizing constant. The density $P^h_{N,Q,\beta}$ can also be written in the form (2) with $\varphi(t) := |t|^\beta \exp\{-h(t)\}$.
Furthermore, let for a probability density P_N on $\mathbb{R}^N$ and $k = 1, 2, \dots,$

$$\rho_N(t_1, \dots, t_k) := \int_{\mathbb{R}^{N-k}} P_N(t_1, \dots, t_k, x_{k+1}, \dots, x_N)\, dx_{k+1} \dots dx_N$$

denote the $k-$th correlation function of P_N. The correlation functions are the marginal densities. The measure $\rho_N(t_1, \dots, t_k)\, dt_1 \dots dt_k$ is called $k-$th correlation measure. Denote by $\rho^{h,k}_{N,Q,\beta}$ the k-th correlation function of $P^h_{N,Q,\beta}$ and by $\rho^k_{N,Q,\beta}$ the k-th correlation function of $P_{N,Q,\beta}$ from (1). Universality of ensembles is usually defined by universality of their correlation functions or measures as many interesting statistics of the ensembles can be expressed in terms of correlation functions. Finally, introduce for a twice differentiable convex function Q the quantity $\alpha_Q := \inf_{t\in\mathbb{R}} Q''(t)$.
The following theorem deals with the global or macroscopic behavior of the ensemble $P^h_{N,Q,\beta}$.

Theorem 1. *Let h be a real analytic and even Schwartz function. Then there exists a constant $\alpha^h \geq 0$ such that for all real analytic, strongly convex and even Q with $\alpha_Q > \alpha^h$, the following holds:*
The first correlation measure $\rho^{h,1}_{N,Q,\beta}$ converges weakly to a compactly supported probability measure $\mu^h_{Q,\beta}$ which has a non-zero and continuous density on the interior of its support. Weak convergence means that for any bounded and continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\lim_{N\to\infty} \int f(t)\, \rho^{h,1}_{N,Q,\beta}(t)\, dt = \int f(t)\, \mu^h_{Q,\beta}(t) dt.$$

Remark 2.

- In general, $\mu^h_{Q,\beta}$ depends on h, i.e. changing the interaction term has an influence on the (limiting) global density of the particles.
- If h is positive semi-definite, then α^h in Theorem 1 may be explicitly chosen as $\alpha^h = \sup_{t\in\mathbb{R}} -h''(t)$.
- For $k = 2, 3, \dots,$ the $k-$th correlation measure converges weakly to the k-fold product $\left(\mu^h_{Q,\beta}\right)^{\otimes k}$. This has been shown in [GV12] for $\beta = 2$ but the same proof goes through for arbitrary $\beta > 0$. As a proof of this statement would lengthen the presentation, we will not pursue it here.
- Note that the dependence of $\mu^h_{Q,\beta}$ on β can be eliminated if the prefactor β is put in front of Q and h.

The next theorem states the local universality in the bulk. We use the notion of universality by Erdős, Yau et al. (see e.g. [EY12] and the references therein). Let G be the Gaussian potential $G(t) := x^2$ and recall that the corresponding limiting measure $\mu_{Q,\beta}$ is the semicircle distribution (with a certain variance depending on β). Recall that under mild assumptions on

Q, there is a measure $\mu_{Q,\beta}$ of compact support which is the weak limit of the first correlation measure of $P_{N,Q,\beta}$. Consider the scaled correlation functions

$$\frac{1}{\mu^h_{Q,\beta}(a)^k}\rho^{h,k}_{N,Q,\beta}\left(a+\frac{t_1}{N\mu^h_{Q,\beta}(a)},\dots,a+\frac{t_k}{N\mu^h_{Q,\beta}(a)}\right), \tag{5}$$

where a is a point with $\mu^h_{Q,\beta}(a)>0$ and $t_1,\dots,t_k$ are contained in an N-independent compact interval. Under this scaling, the local density around a will be asymptotically one, in particular independent of a. For $N\to\infty$, $h=0$ and $Q=G$, the limit of (5) exists and has been described in terms of a stochastic process in [VV09]. As for general β no nice formula for this limit is known, we state the following theorem as universality result, comparing the local correlations of $P^h_{N,Q,\beta}$ with those of the Gaussian β-Ensemble $P_{N,G,\beta}$.

Theorem 3. *Let h and Q satisfy the assumptions of Theorem 1. Let $0<\xi\le 1/2$ and set $s_N:=N^{-1+\xi}$. Then for $k=1,2,\dots,$ we have for any a in the interior of the support of $\mu^h_{Q,\beta}$, any a' in the interior of the support of the semicircle law $\mu_{G,\beta}$ and any smooth function $f:\mathbb{R}^k\longrightarrow\mathbb{R}$ with compact support*

$$\begin{aligned}\lim_{N\to\infty}\int f(t_1,\dots,t_k)\Bigg[&\int_{a-s_N}^{a+s_N}\frac{1}{\mu^h_{Q,\beta}(a)^k}\rho^{h,k}_{N,Q,\beta}\left(u+\frac{t_1}{N\mu^h_{Q,\beta}(a)},\dots,u+\frac{t_k}{N\mu^h_{Q,\beta}(a)}\right)\frac{du}{2s_N}\\ &-\int_{a'-s_N}^{a'+s_N}\frac{1}{\mu_{G,\beta}(a')^k}\rho^{k}_{N,G,\beta}\left(u+\frac{t_1}{N\mu_{G,\beta}(a')},\dots,u+\frac{t_k}{N\mu_{G,\beta}(a')}\right)\frac{du}{2s_N}\Bigg]dt_1\dots dt_k\\ &=0.\end{aligned}$$

Remark 4.

- If the inner integrations were not present, the convergence in Theorem 3 would be vague convergence of the scaled correlation measures. Here an additional small (uniform) average around the points a and a' is performed.
- If h is positive semi-definite, then α^h in Theorem 3 may be explicitly chosen as $\alpha^h=\sup_{t\in\mathbb{R}}-h''(t)$.
- The choice of the Gaussian β-Ensemble $P_{N,G,\beta}$ is just for definiteness, in fact any other ensemble belonging to the same universality class could be chosen. So far, these are known to be basically all $P_{N,Q,\beta}$ with the same β and real analytic Q which leads to a limiting measure $\mu_{Q,\beta}$ of connected support [BEY12].

These results should be compared to those of [GV12]. There we could show for $\beta=2$ under the same conditions on Q and h a much stronger type of convergence as in Theorem 3. We proved in [GV12]

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{\mu^h_{Q,\beta=2}(a)^k}\rho^{h,k}_{N,Q,\beta=2}\left(a+\frac{t_1}{N\mu^h_{Q,\beta=2}(a)},\dots,a+\frac{t_k}{N\mu^h_{Q,\beta=2}(a)}\right)\\ &=\det\left[\frac{\sin\left(\pi(t_i-t_j)\right)}{\pi(t_i-t_j)}\right]_{1\le i,j\le k}\end{aligned}$$

uniformly in $t_1, \dots, t_k$ from any compact subset of $\mathbb{R}^k$ and uniformly in the point a from any compact proper subset of the support of $\mu^h_{Q,\beta=2}$. This locally uniform convergence of the marginal densities was inherited from strong results on universality of unitary invariant (i.e. $\beta = 2$) ensembles (cf. [LL08]). In order to apply these results, we developed a method to express the correlation functions of the model $P^h_{N,Q,\beta=2}$ as a probabilistic mixture of unitary invariant ensembles with potential $V + f/N$, where V was fixed and f was random. However, this representation was only possible for negative semi-definite h and an argument involving complex analysis had to be used to extend the universality for more general h.
So far, the local relaxation flow approach due to Erdős, Schlein and Yau (refined by others) [ESY11] and applied to β-Ensembles by Bourgade, Erdős and Yau [BEY11, BEY12] is the only method for showing bulk universality for general β-Ensembles. A remark on some crucial points of this method is included in Section 4. Their approach actually addresses universality of gap distributions which implies the weaker form of universality of the correlation measures as stated in Theorem 3. As we use their method, we obtain the same form of convergence. If other sufficiently general universality results on β-Ensembles yielding stronger types of convergence were available, the method of [GV12] could be used to prove Theorem 3 with stronger forms of convergence. One advantage of the local relaxation flow approach is the possibility to compare local statistics of eigenvalue ensembles and other, not necessary spectral ensembles, directly. This allows us to give a short proof of Theorem 3.
Similar in both [GV12] and the present work is the identification of the global behavior of $P^h_{N,Q,\beta}$, e.g. Theorem 1. The limiting measure is identified in Section 2 as a (unique) solution of a certain recursive equation. Using this measure, from Q and h a new potential V is defined and to $P^h_{N,Q,\beta}$ an invariant ensemble $P_{N,V,\beta}$ is associated for which $P^h_{N,Q,\beta} \sim \exp\{\mathcal{U}\} P_{N,V,\beta}$ holds. In Section 3, a concentration of measure result is shown for $\mathcal{U}$, furthermore Theorem 1 is proved. Section 4 contains the proof of Theorem 3 via the local relaxation flow approach. We finish with an appendix on some results about equilibrium measures which are needed in Section 2.

2. The associated invariant ensemble

In this section we determine the limiting measure for our particle system and use this to construct an ensemble of eigenvalues with the same global and local behaviour. This is analogous to [GV12]. In order to keep the presentation on the new features of $P^h_{N,Q,\beta}$ self-contained, we include it here.
Let $\beta > 0$, h be a continuous even function, Q a strictly convex symmetric function and assume that

$$P^h_{N,Q,\beta}(x) := \frac{1}{Z^h_{N,Q,\beta}} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \, e^{-N \sum_{j=1}^N Q(x_j) - \sum_{i<j} h(x_i - x_j)}, \tag{6}$$

defines the density of a probability measure on $\mathbb{R}^N$, where

$$Z^h_{N,Q,\beta} := \int_{\mathbb{R}^N} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \, e^{-N \sum_{j=1}^N Q(x_j) - \sum_{i<j} h(x_i - x_j)} dx$$

denotes the normalizing constant. We will use the notation

$$f_\mu(s) := \int f(t-s)d\mu(t), \quad f_{\mu\mu} := \int\int f(t-s)d\mu(t)d\nu(s) \tag{7}$$

for an even function $f : \mathbb{R} \longrightarrow \mathbb{R}$ of sufficient integrability. For the statement of the next lemma, $\mathcal{M}_c^1$ will denote the set of compactly supported (Borel) probability measures on $\mathbb{R}$. Recall that the unique minimizer of the functional

$$I_{Q,\beta}(\mu) := \int Q(t)d\mu(t) + \frac{\beta}{2}\int\int \log|s-t|^{-1} d\mu(s)d\mu(t)$$

is called equilibrium measure to the external field Q and $\beta > 0$.

Lemma 5. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be even, twice differentiable, bounded and such that $h''(t) \geq -\alpha_Q$ for all t. Define $T_{h,\beta} : \mathcal{M}_c^1 \longrightarrow \mathcal{M}_c^1$, $T_{h,\beta}(\mu)$ as the equilibrium measure to the external field $t \mapsto Q(t) + h_\mu(t)$ and β.*
Then $T_{h,\beta}$ has a fixed point, i.e. there exists a probability measure $\mu_{Q,\beta}^h$ which is the equilibrium measure to the external field $t \mapsto Q(t) + \int h(t-s)d\mu_{Q,\beta}^h(s)$.

Proof. Without loss of generality we only consider the standard case $\beta = 2$ and omit the index β, the general case can be reduced to this case by considering Q/β and h/β.
Recall Schauder's Fixed Point Theorem which states that each continuous mapping $T : C \longrightarrow C$ of a compact, convex and non-empty subset C of a Hausdorff topological vector space has a fixed point.
Consider the Hausdorff topological vector space $\mathcal{M}(K)$ of all signed finite Borel measures on some compact interval K of $\mathbb{R}$, equipped with the topology of vague convergence. The subset $\mathcal{M}_s^1(K)$ of all symmetric Borel probability measures on K is non-empty, convex and compact by Helly's Selection Theorem.
The first step in the proof is to show that it follows from $h''(t) \geq -\alpha_Q$ and the boundedness of h, that the support of the equilibrium measure to the external field $Q(t) + h_\mu(t)$ is for all μ included in a compact interval of $\mathbb{R}$. By Theorem A.6 (in the appendix), the support of the equilibrium measure for $Q(t) + h_\mu(t)$ is the smallest compact set K of positive capacity maximizing the functional

$$\begin{aligned} K \mapsto F_{Q+h_\mu}(K) &= \log \operatorname{cap}(K) - 2\int Q(t)d\omega_K(t) - 2\int h_\mu(t)d\omega_K(t) \\ &= F_Q(K) - 2\int h_\mu(t)d\omega_K(t). \end{aligned} \tag{8}$$

Choosing for K the support suppμ_Q of the equilibrium measure μ_Q, we get using $|h_\mu| \leq \|h\|_\infty$ the simple inequality

$$F_{Q+h_\mu}(\text{supp}\mu_Q) \geq F_Q(\text{supp}\mu_Q) - 2\|h\|_\infty \in \mathbb{R}. \tag{9}$$

It is easy to see that we have $(h_\mu)'' = (h'')_\mu$. By the condition $h''(t) \geq -\alpha_Q$, $Q(t) + h_\mu(t)$ is convex for each compactly supported μ. By Theorem A.6, the support of $T_h(\mu)$ is a symmetric interval, say $[-l_\mu, l_\mu]$. Using Lemma A.1, we can rewrite (8) for an arbitrary

symmetric interval $[-l,l]$ as

$$F_{Q+h_\mu}([-l,l]) = \log(l/2) - 2\int_{-l}^{l} Q(t)\frac{1}{\pi\sqrt{l^2-t^2}}dt - 2\int_{-l}^{l} h_\mu(t)\frac{1}{\pi\sqrt{l^2-t^2}}dt. \tag{10}$$

Since Q is strictly convex and symmetric, we have $Q(t) \geq \alpha_Q t^2 + C$ for some $C \in \mathbb{R}$ and (10) implies (using that the variance of $\omega_{[-l,l]}$ is $l^2/2$) the inequality

$$F_{Q+h_\mu}([-l,l]) \leq \log(l/2) - \alpha_Q l^2 - C + 2\|h\|_\infty, \tag{11}$$

which holds for any μ. From (9) and (11) we see that

$$F_{Q+h_\mu}(\mathrm{supp}\mu_Q) > F_{Q+h_\mu}([-l,l]),$$

for all $l > L$ for some μ-independent L. Hence $l_\mu \leq L$ for all compactly supported μ.
Thus T_h maps the set $\mathcal{M}_s^1(K)$ into itself, if K is chosen large enough. It remains to show continuity. We will show that T_h maps converging sequences to converging sequences. Let $(\mu_n)_n \subset \mathcal{M}^1(K)$ be a sequence converging vaguely, or equivalently, weakly to a probability measure μ. Denote $T_h(\mu_n) =: \nu_n$. Then the sequence of external fields $V_n(t) := Q(t) + h_{\mu_n}(t)$ converges pointwise to $V(t) := Q(t) + h_\mu(t)$. Since by Theorem A.4, the equilibrium measure does not depend on values of the external field outside of its support, we can assume this convergence to be uniform. Indeed, as h' is bounded on the compact set K by some constant C, we have $\left|h'_{\mu_n}\right| \leq C$. It follows that the sequence $(h_{\mu_n})_n$ is uniformly Lipschitz and hence equicontinuous. Thus the sequence $(V_n)_n$ is also equicontinuous. Since their domain is a compact and V_n converges pointwise, the equicontinuity implies uniform convergence by Arzela-Ascoli's Theorem.
Because all ν_n are supported on the same compact set, it follows that $(\nu_n)_n$ is tight and hence has a weakly converging subsequence $(\nu_{n_m})_m$. We will prove that this limit measure, say ν', is in fact $\nu = T_h(\mu)$, the measure belonging the external field V, and does not depend on the particular subsequence. It follows that the sequence $(\nu_n)_n$ converges to ν weakly. From the uniform convergence of V_n towards V it follows by Theorem A.5 1. that

$$U^{\nu_{n_m}}(s) = \int \log|t-s|^{-1}\,d\nu_{n_m}(t)$$

converges uniformly (on $\mathbb{C}$) towards $U^\nu(s) := \int \log|t-s|^{-1}\,d\nu(t)$. On the other hand, by Theorem A.5 we have for all $s \in \mathbb{C}$ except a set of zero capacity

$$\lim_{m\to\infty} U^{\nu_{n_m}}(s) = U^{\nu'}(s) = \int \log|t-s|^{-1}\,d\nu'(t),$$

which yields $U^\nu(s) = U^{\nu'}(s)$ almost everywhere on $\mathbb{C}$ and hence $\nu = \nu'$ by Theorem A.5, implying that the sequence $(\nu_n)_n$ converges weakly to ν. □

Remark 6 (Uniqueness). Uniqueness of the fixed point will follow for the class of ensembles from Theorem 1. For those ensembles we will prove that the first correlation measure converges weakly to any fixed point, thereby showing uniqueness.

We will now use Lemma 5 to construct an associated invariant ensemble. Let h be as in Lemma 5. Choose a fixed point $\mu_{Q,\beta}^h$ as in Lemma 5. We will stick to this measure from now

on and write μ instead of $\mu^h_{Q,\beta}$. Using notation (7), we make the Hoeffding type decomposition

$$\begin{aligned}
&\sum_{i<j} h(x_i - x_j) \\
&= -\frac{N^2}{2} h_{\mu\mu} - \frac{N}{2} h(0) + N \sum_{j=1}^N h_\mu(x_j) + \frac{1}{2}\Big(\sum_{i,j=1}^N h(x_i - x_j) - [h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}] \Big) \\
&= -\frac{N^2}{2} h_{\mu\mu} - \frac{N}{2} h(0) + N \sum_{j=1}^N h_\mu(x_j) - \mathcal{U}(x), \qquad \text{where} \qquad (12)
\end{aligned}$$

$$\mathcal{U}(x) := -\frac{1}{2}\Big(\sum_{i,j=1}^N h(x_i - x_j) - [h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}] \Big). \tag{13}$$

Now we can rewrite $P^h_{N,Q,\beta}$ as

$$P^h_{N,Q,\beta}(x) = \frac{1}{Z_{N,V,\beta,\mathcal{U}}} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \, e^{-N \sum_{j=1}^N V(x_j) + \mathcal{U}(x)}, \tag{14}$$

where we defined the external field

$$V(t) := Q(t) + h_\mu(t)$$

and absorbed the constant $\exp\{-(N^2/2)h_{\mu\mu} - (N/2)h(0)\}$ into the new normalizing constant $Z_{N,V,\beta,\mathcal{U}}$. We will often use this representation of $P^h_{N,Q,\beta}$. The proofs of Theorems 1 and 3 rely on comparison of $P^h_{N,Q,\beta}$ with the invariant ensemble

$$P_{N,V,\beta}(x) = \frac{1}{Z_{N,V,\beta}} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \, e^{-N \sum_{j=1}^N V(x_j)}. \tag{15}$$

3. Concentration of $\mathcal{U}$

This section is similar to Section 4 in [GV12] except for the proof of Theorem 1. As several arguments of this section are needed later, we include it in the presentation.
A key tool will be the following well-known concentration of measure inequality ([AGZ10, Section 4.4]).

Theorem 7. *Let Q be two times differentiable with $Q'' \ge c > 0$. Then we have for any Lipschitz function f and any $\varepsilon > 0$*

$$P_{N,Q,\beta}\Big(\Big| \sum_{j=1}^N f(x_j) - \mathbb{E}_{N,Q,\beta} \sum_{j=1}^N f(x_j) \Big| > \varepsilon \Big) \le 2 \exp\Big\{ -\frac{c\varepsilon 2}{2\,|f|_{\mathcal{L}}\,2} \Big\} \qquad \textit{and}$$

$$\mathbb{E}_{N,Q,\beta} \exp\Big\{ \varepsilon \Big(\sum_{j=1}^N f(x_j) - \mathbb{E}_{N,Q,\beta} \sum_{j=1}^N f(x_j) \Big) \Big\} \le \exp\Big\{ \frac{\varepsilon^2\, |f|^2_{\mathcal{L}}}{2c} \Big\},$$

where we denote the Lipschitz constant of f by $|f|_{\mathcal{L}}$.

The following is a special case of a result in [Shc11] (see also [KS10]). By $\mu_{Q,\beta}$ we will denote the equilibrium measure to Q (and β).

Proposition 8. *Let Q be a convex external field on $\mathbb{R}$ which is real analytic in a neighborhood of* $\operatorname{supp}(\mu_{Q,\beta})$*. Let f be a function whose third derivative is bounded on a neighborhood of* $\operatorname{supp}(\mu_{Q,\beta})$*. Then*

$$\big|\mathbb{E}_{N,Q,\beta}\sum_{j=1}^{N} f(x_j) - N\int f d\mu_{Q,\beta}\big| \leq C(\|f\|_\infty + \|f^{(3)}\|_\infty),$$

where C does not depend on N or f and $\|\cdot\|_\infty$ denotes the bound on the neighborhood of $\operatorname{supp}(\mu_{Q,\beta})$*.*

Theorem 7 and Proposition 8 yield immediately

Corollary 9. *Let Q be a real analytic external field with $Q'' \geq c > 0$. Then for any Lipschitz function f whose third derivative is bounded on a neighborhood of* $\operatorname{supp}(\mu_{Q,\beta})$*, we have for any $\varepsilon > 0$*

$$\mathbb{E}_{N,Q,\beta}\exp\big\{\varepsilon\big(\sum_{j=1}^{N} f(x_j) - N\int f(t)d\mu_{Q,\beta}(t)\big)\big\} \leq \exp\big\{\frac{\varepsilon^2\,|f|_{\mathcal{L}}^2}{2c} + \varepsilon C(\|f\|_\infty + \|f^{(3)}\|_\infty)\big\}.$$

The idea is to reduce concentration of the bivariate statistic $\mathcal{U}$ to concentration of linear statistics. To this end, we give an alternative representation of $\mathcal{U}$ using Fourier techniques. A similar idea is used in [LP08].

Lemma 10. *We have*

$$\mathcal{U}(x) = -\frac{1}{2\sqrt{2\pi}}\int \left|\mathring{u}_N(t,x)\right|^2\,\widehat{h}(t)dt, \qquad \textit{where}$$

$$\mathring{u}_N(t,x) := \sum_{j=1}^{N}\cos(tx_j) - N\int\cos(ts)d\mu(s) + \sqrt{-1}\sum_{j=1}^{N}\sin(tx_j),\ \widehat{h}(t) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-its}h(s)ds.$$

Proof. Recall from (13) that

$$\mathcal{U}(x) = -\frac{1}{2}\big(\sum_{i,j=1}^{N} h(x_i - x_j) - [h_\mu(x_i) + h_\mu(x_j) - h_{\mu\mu}]\big).\quad \text{Note that}$$

$$\frac{1}{2}\sum_{j,k} h(x_j - x_k) = \frac{1}{2\sqrt{2\pi}}\int\sum_{j,k} e^{i(x_j - x_k)t}\,\widehat{h}(t)dt = \frac{1}{2\sqrt{2\pi}}\int |u_N(t,x)|^2\,\widehat{h}(t)dt,$$

with $u_N(t,x) := \sum_{j=1}^{N} e^{itx_j}$. Writing $\mathring{u}_N(t,x) := u_N(t,x) - N\int e^{its}d\mu(s)$, it is not hard to check that

$$\mathcal{U}(x) = -\frac{1}{2\sqrt{2\pi}}\int\left|\mathring{u}_N(t,x)\right|^2\,\widehat{h}(t)dt. \tag{16}$$

□

A trivial but useful observation is

$$\mathbb{E}^h_{N,Q,\beta}f(x) = (Z_{N,V,\beta}/Z_{N,V,\beta,\mathcal{U}})\mathbb{E}_{N,V,\beta}f(x)e^{\mathcal{U}(x)}.$$

The next proposition establishes concentration of $\mathcal{U}$.

Proposition 11. *If the constant α_Q is large enough, then there exist constants $C_1, C_2 > 0$ such that for all N*

$$0 < C_1 \leq Z_{N,V,\beta,\mathcal{U}}/Z_{N,V,\beta} = \mathbb{E}_{N,V,\beta} \exp\big\{\mathcal{U}(x)\big\} \leq C_2.$$

Proof. By Jensen's inequality we get

$$\mathbb{E}_{N,V,\beta} \exp\big\{\mathcal{U}(x)\big\} \geq \exp\big\{\mathbb{E}_{N,V,\beta}\mathcal{U}(x)\big\}.$$

Using Lemma 10 we show that the expectation of $\mathcal{U}$ is bounded in N. Fubini's Theorem gives

$$\begin{aligned}
-\mathbb{E}_{N,V,\beta}\mathcal{U}(x) &= \frac{1}{2\sqrt{2\pi}} \int \mathbb{E}_{N,V,\beta} \Big|\mathring{u}_N(t,x)\Big|^2 \ \widehat{h}(t) dt \\
&= \frac{1}{2\sqrt{2\pi}} \int \left(\mathbb{E}_{N,V,\beta} \Big| \sum_{j=1}^N \cos(tx_j) - N \int \cos(ts) d\mu(s) \Big|^2 + \mathbb{E}_{N,V,\beta} \Big|\sum_{j=1}^N \sin(tx_j)\Big|^2 \right) \widehat{h}(t) dt.
\end{aligned}$$

By Corollary 9, the terms in the parenthesis are bounded by a polynomial in t, as $|\cos(t\cdot)|_{\mathcal{L}}, |\sin(t\cdot)|_{\mathcal{L}} \leq t$ and $\|\cos(t\cdot)^{(3)}\|_\infty, \|\sin(t\cdot)^{(3)}\|_\infty \leq Ct^3$. Hence, $\widehat{h}$ being a Schwartz function, we have $\mathbb{E}_{N,V,\beta}\mathcal{U}(x) \geq -C'$ for some $C' > 0$. Thus the lower bound follows choosing $C_1 := \exp(-C')$.

For the upper bound recall that since h is even, $\widehat{h}$ is real-valued. Define $\widehat{h}_+(y) := \max\{0, \widehat{h}(y)\}$ and $\widehat{h}_-(y) := \max\{0, -\widehat{h}(y)\}$ such that $\widehat{h} = \widehat{h}_+ - \widehat{h}_-$. For $\widehat{h}_- = 0$, which corresponds to the case of a positive semi-definite h, there is nothing to prove, so assume that $\widehat{h}_- \neq 0$. Introducing $H_- := \big(\widehat{h}_-\big)^{1/2} \geq 0$, Jensen's inequality and Tonelli's Theorem give

$$\begin{aligned}
&\mathbb{E}_{N,V,\beta} \exp\Big\{ -(2\sqrt{2\pi})^{-1} \int \widehat{h}(t) \big|\mathring{u}_N(t,x)\big|^2 dt \Big\} \leq \mathbb{E}_{N,V,\beta} \exp\Big\{ (2\sqrt{2\pi})^{-1} \int H_-(t)^2 \ \big|\mathring{u}_N(t,x)\big|^2 dt \Big\} \\
&= \mathbb{E}_{N,V,\beta} \exp\Big\{ (2\sqrt{2\pi})^{-1} \big\|H_-\big\|_{L^1} \int \big(H_-(t)/\big\|H_-\big\|_{L^1}\big) H_-(t) \big|\mathring{u}_N(t,x)\big|^2 dt \Big\} \\
&\leq \int \big(H_-(t)/\big\|H_-\big\|_{L^1}\big) \mathbb{E}_{N,V,\beta} \exp\Big\{ (2\sqrt{2\pi})^{-1} \big\|H_-\big\|_{L^1} H_-(t) \big|\mathring{u}_N(t,x)\big|^2 \Big\} dt. \qquad (17)
\end{aligned}$$

Abbreviating $K_h := (2\sqrt{2\pi})^{-1} \big\|H_-\big\|_{L^1}$ and using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
&\mathbb{E}_{N,V,\beta} \exp\Big\{ K_h H_-(t) \big|\mathring{u}_N(t,x)\big|^2 \Big\} && (18)\\
&\leq \mathbb{E}^{1/2}_{N,V,\beta} \exp\Big\{ 2K_h H_-(t) \Big| \sum_{j=1}^N \cos(tx_j) - N \int \cos(ts) d\mu(s) \Big|^2 \Big\} && (19)\\
&\qquad \times \mathbb{E}^{1/2}_{N,V,\beta} \exp\Big\{ 2K_h H_-(t) \ \Big| \sum_{j=1}^N \sin(tx_j) \Big|^2 \Big\}. && (20)
\end{aligned}$$

Considering only (19), we have by Corollary 9 for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}_{N,V,\beta} \exp\Big\{ \varepsilon \cdot \sqrt{2K_h H_-(t)} \Big(\sum_{j=1}^N \cos(tx_j) - N \int \cos(ts) d\mu(s) \Big) \Big\} \\
&\leq \exp\Big\{ \varepsilon^2 \cdot 2K_h H_-(t) t^2 (2\alpha_V)^{-1} + \varepsilon \sqrt{2K_h H_-(t)} C(1+t^3) \Big\}, \qquad (21)
\end{aligned}$$

where $\alpha_V := \min_t V''(t) > 0$ and C does not depend on t or N. For α_Q large enough (hence α_V large enough), we have $2K_h H_-(t)t^2(2\alpha_V)^{-1} < 1/4$ for all t. Since $H_-(t) = \widehat{h}_-^{1/2}(t)$ is decaying rapidly, $\sqrt{2K_h H_-(t)}C(1+t^3)$ is bounded in t. Summarizing, if α_Q is large enough, we can bound (21) by

$$\exp\{c\varepsilon^2 + \varepsilon C\}$$

with $0 < c < 1/4$ and c, C do not depend on N or t. Thus we have

$$\begin{aligned}
&\mathbb{E}_{N,V,\beta} \exp\Big\{2K_h H_-(t)\Big|\sum_{j=1}^N \cos(tx_j) - N\int \cos(ts)d\mu(s)\Big|^2\Big\} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big\{\varepsilon \cdot \sqrt{2K_h H_-(t)}\Big(\sum_{j=1}^N \cos(tx_j) - N\int \cos(ts)d\mu(s)\Big)\Big\} \exp\{-\varepsilon^2/4\}d\varepsilon \\
&\le \frac{1}{2\pi}\int_{\mathbb{R}} \exp\{c\varepsilon^2 + \varepsilon C\}\exp\{-\varepsilon^2/4\}d\varepsilon \le C' \quad \text{ for some } C'.
\end{aligned}$$

We conclude that (18) is bounded in N as long as α_Q is large enough. Finally, it follows from (17) that

$$\mathbb{E}_{N,V,\beta} \exp\big\{-\int \widehat{h}(t)\big|\mathring{u}_N(t,x)\big|^2 dt\big\} \le C$$

for some constant $C > 0$ independent of N. This proves the upper bound. $\square$

Remark 12. The proof of Proposition 11 actually shows that for each $\lambda > 0$ there is a threshold $\alpha^h(\lambda) > 0$ and constants C_1, C_2 (depending on λ and α^h) such that

$$0 < C_1 < \mathbb{E}_{N,V,\beta}\exp\{\lambda\mathcal{U}(x)\} \le C_2, \quad \text{ if } \alpha_Q \ge \alpha^h(\lambda).$$

Proof of Theorem 1. It remains to show that the fixed point μ, whose existence was obtained in Lemma 5, is unique and indeed the limit of the first correlation function. We consider a Lipschitz function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with three continuous derivatives and estimate for any $\varepsilon > 0$

$$P^h_{N,Q,\beta}(|N^{-1}\sum_{j=1}^N f(x_j) - \int f d\mu| > \varepsilon) = (Z_{N,V,\beta}/Z_{N,V,\beta,\mathcal{U}})\mathbb{E}_{N,V,\beta} e^{\mathcal{U}(x)} \mathbb{1}_{\{|N^{-1}\sum_{j=1}^N f(x_j) - \int f d\mu| > \varepsilon\}}.$$

By Hölder's inequality and Remark 12, we have

$$P^h_{N,Q,\beta}(|N^{-1}\sum_{j=1}^N f(x_j) - \int f d\mu| > \varepsilon) \le C\big(P_{N,V,\beta}(|N^{-1}\sum_{j=1}^N f(x_j) - \int f d\mu| > \varepsilon)\big)^c$$

for some $c, C > 0$. By Corollary 9, this last probability converges for any $\varepsilon > 0$ to 0 exponentially fast as $N \to \infty$. We conclude with Lebesgue's Theorem that

$$\lim_{N\to\infty} \mathbb{E}^h_{N,Q,\beta}|N^{-1}\sum_{j=1}^N f(x_j) - \int f d\mu| = 0 \text{ and hence } \lim_{N\to\infty} \mathbb{E}^h_{N,Q,\beta} N^{-1}\sum_{j=1}^N f(x_j) = \int f d\mu.$$

As convergence for smooth Lipschitz functions determines weak convergence, the weak convergence of the first correlation measure follows. As the limit of weak convergence is unique,

this shows uniqueness of the fixed point in Lemma 5. The existence and positivity of the continuous density of μ is clear by Theorem A.7 as V is real-analytic and strictly convex. $\square$

4. Proof of Theorem 3

In this section we use the local relaxation flow approach developed by Erdős, Yau, Schlein et. al. to establish universality of the local bulk correlations. First we introduce some notation from [BEY11].

Let k be fixed. Let $G : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a smooth function with compact support and $m = (m_1, \dots, m_k)$ with m_j being positive integers. Define

$$G_{i,m}(x) := G(N(x_i - x_{i+m_1}), \dots, N(x_{i+m_{k-1}} - x_{i+m_k})).$$

The Dirichlet form of a smooth test function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ w.r.t. a probability measure $d\omega$ on $\mathbb{R}^N$ is defined as

$$D_\omega(f) := \frac{1}{2N} \sum_{j=1}^{N} \int \left(\partial_{x_j} f\right)^2 d\omega.$$

Let f be a probability density function w.r.t. $d\omega$. The (relative) entropy of f w.r.t. $d\omega$ is defined as

$$S_\omega(f) := \int f \log f d\omega.$$

We will use the following general theorem.

Proposition 13. [BEY11, Lemma 5.9] *Let $G : \mathbb{R}^k \longrightarrow \mathbb{R}$ be bounded and of compact support. Let $d\omega$ be a probability measure on $\{x : x_1 < x_2 < \cdots < x_N\} \subset \mathbb{R}^N$ given by*

$$d\omega = \frac{1}{Z} e^{-\beta N \widehat{\mathcal{H}}(x)} dx, \quad \widehat{\mathcal{H}}(x) = \mathcal{H}_0(x) - \frac{1}{N} \sum_{i<j} \log |x_j - x_i|$$

with the property that $\nabla^2 \mathcal{H}_0 \geq \tau^{-1}$ holds for some positive constant τ. Let $q d\omega$ be another probability measure with smooth density q. Let $J \subset \{1, 2, \dots, N - m_k - 1\}$ be a set of indices. Then for any $\varepsilon_1 > 0$ we have

$$\left| \frac{1}{|J|} \sum_{i \in J} \int G_{i,m}\, q d\omega - \frac{1}{|J|} \sum_{i \in J} \int G_{i,m} d\omega \right| \leq C \sqrt{N^{\varepsilon_1} \frac{D_\omega(\sqrt{q})\tau}{|J|}} + C\sqrt{S_\omega(q)} e^{-cN^{\varepsilon_1}}.$$

In our application we will choose $d\omega = P_{N,V,\beta}$ and $q = (Z_{N,V,\beta}/Z_{N,V,\beta,\mathcal{U}}) \exp\{\mathcal{U}\}$. If α_Q is large enough, then V is strongly convex, hence $\tau = 1/\alpha_V$. By the symmetry of $P^h_{N,Q,\beta}$ and $P_{N,V,\beta}$, it is equivalent to restrict the measure to the simplex $\{x : x_1 < x_2 < \cdots < x_N\}$ and multiply by $N!$. From [EY12, Theorem 2.3] we have that

$$S_\omega(q) \leq C D_\omega(\sqrt{q}).$$

It is thus sufficient to prove that $D_\omega(\sqrt{q})$ is bounded in N as J will be chosen such that $|J| \sim N$ in order to identify the bulk correlations.

Remark 14 (On the local relaxation flow approach). To briefly explain the essence of this method due to Erdős, Schlein, Yau and others (see e.g. [EY12] for references and a complete review), let us consider two measures as in Proposition 13, $d\omega$ and $qd\omega$ and their statistics $\int gd\omega$ and $\int g\,qd\omega$ for some test function g. Assume that one can define a Markov process on $\mathbb{R}^N$ in terms of the Dirichlet form D_ω (or the formal generator $L_N := \frac{1}{2N}\Delta - \frac{1}{2}(\nabla\widehat{\mathcal{H}})\nabla)$, having $d\omega$ as stationary distribution. Assume that the process has the initial distribution $qd\omega$ and denote the evolution of the density w.r.t. $d\omega$ by $(f_t)_{t\geq 0}, f_0 = q, f_\infty = 1$. Then one can write

$$\int g\,qd\omega - \int gd\omega = \Big(\int g\,qd\omega - \int g\,f_t d\omega\Big) + \Big(\int g\,f_t d\omega - \int gd\omega\Big),$$

which corresponds to running the process up to time t. If the process is ergodic and the time t is large enough, $\int g\,f_t d\omega$ will be close to the equilibrium $\int g\,d\omega$. If this t is still "small", i.e. the convergence to the stationary distribution is fast, then the distance between $\int g\,qd\omega$ and $\int g\,f_t d\omega$ should be not too big. These distances are measured in terms of Dirichlet form and entropy of $d\omega$. These estimates are due to the Bakry-Emery method which yields ergodicity or *relaxation* making use of the strict convexity of the Hamiltonian, i.e. of the bound $\nabla^2\mathcal{H}_0 \geq \tau^{-1}$. It turns out that the constant τ is the time scale for the relaxation to equilibrium, meaning that e.g. $S_\omega(f_t) \leq e^{-t/\tau}S_\omega(f_0)$. Here we tacitly used that the logarithmic part of the Hamiltonian $\widehat{\mathcal{H}}$ is convex, therefore does not increase the relaxation time. However, one crucial observation is that from the trivial bound

$$\langle v, \nabla^2\widehat{\mathcal{H}}(x)v\rangle \geq \frac{1}{\tau}\|v\|^2 + \frac{1}{N}\sum_{i<j}\frac{(v_i - v_j)^2}{(x_i - x_j)^2}$$

one can infer that the relaxation is much faster in the directions $(v_i - v_j)$ provided that x_i and x_j are close. Indeed, the mean distance between neighboring eigenvalues is of order $1/N$, hence the convexity bound for the Hamiltonian should be locally of order N, therefore yielding a time to the local equilibrium of order $1/N$ whereas the time to the global equilibrium is of order 1. This informal reasoning can be captured by choosing test functions like $G_{i,m}$ which depend only on eigenvalue differences in the local scaling (i.e. multiplied by N) and vanish whenever two eigenvalues are not close to each other. By exploiting these features of $G_{i,m}$ and some estimates, one arrives at Proposition 13. For arbitrary test functions g, one would get basically the same estimate except for the quantity $|J| \sim N$ which divides $D_\omega(\sqrt{q})$.
One problem with this idea is that the existence of the process associated to the Dirichlet form is not clear for $\beta \in (0,1)$. For $\beta \geq 1$, the repulsion is strong enough to prevent collision between the eigenvalues but for $\beta < 1$ the probability of explosion is positive. This problem was overcome in [EKYY12] by smoothing the singular logarithmic term and using the approach above for the corresponding process.
In most applications the method sketched above was not directly used to prove bulk universality because the quantity $D_\omega(\sqrt{q})/|J|$ could not be estimated to decay as $N \to \infty$. To overcome this difficulty, $d\omega$ was modified by adding another strongly convex potential to the Hamiltonian which confined the eigenvalues x_j near their classical positions γ_j which are defined by the relation $j = N\int_{-\infty}^{\gamma_j} d\mu$, where μ is the equilibrium measure of the ensemble. This additional convexification was shown to slow down the relaxation time to the global equilibrium τ such that $\tau \sim N^{-\varepsilon}$ which was then enough to prove bulk universality. To show that

the local statistics are not influenced by this modification of the ensemble, a strong bound on the *rigidity* of the eigenvalues is needed, which means informally speaking concentration of the eigenvalues around their classical positions. This strategy of proving bulk universality has proven very useful whenever strong bounds on the rigidity of the eigenvalues are available. Much effort in the works of Erdős, Yau et al. has been made to establish these bounds.
In our application, no additional convexification of the Hamiltonian is needed as we can effectively estimate the Dirichlet form in our case. This is due to the fact that $d\omega$ and $qd\omega$ have the same global limit and we have concentration of $\mathcal{U}$ under $d\omega = P_{N,V,\beta}$.

Proposition 15. *Let $D_{N,V,\beta}$ denote the Dirichlet form w.r.t. $P_{N,V,\beta}$ and $q = (Z_{N,V,\beta}/Z_{N,V,\beta,\mathcal{U}})\exp\{\mathcal{U}\}$. Then there is a constant C such that we have for α_Q large enough*

$$D_{N,V,\beta}(\sqrt{q}) \leq C \quad \textit{for all } N.$$

Proof. The ratio $Z_{N,V,\beta}/Z_{N,V,\beta,\mathcal{U}}$ is bounded by Proposition 11 and therefore negligible. We have by Hölder's inequality for $\varepsilon > 0$

$$D_{N,V,\beta}(\sqrt{q}) \leq C\frac{1}{2N}\sum_{l=1}^{N}\mathbb{E}_{N,V,\beta}\big(\partial_{x_l}\exp\{\frac{1}{2}\mathcal{U}(x)\}\big)^2 = C\frac{1}{8N}\sum_{l=1}^{N}\mathbb{E}_{N,V,\beta}\exp\{\mathcal{U}(x)\}\big(\partial_{x_l}\mathcal{U}(x)\big)^2$$
$$\leq C\big(\mathbb{E}_{N,V,\beta}\exp\{(1+\varepsilon)\mathcal{U}(x)\}\big)^{1/\varepsilon}\frac{1}{8N}\sum_{l=1}^{N}\big(\mathbb{E}_{N,V,\beta}\big|\partial_{x_l}\mathcal{U}(x)\big|^{2(\varepsilon+1)/\varepsilon}\big)^{\varepsilon/(\varepsilon+1)}.$$

Again by Proposition 11, $\big(\mathbb{E}_{N,V,\beta}\exp\{(1+\varepsilon)\mathcal{U}(x)\}\big)^{1/\varepsilon}$ is bounded in N. In order to bound the second term, recall that

$$\mathcal{U}(x) = -\frac{1}{2\sqrt{2\pi}}\int\big(\big|\sum_{j=1}^{N}\cos(tx_j) - N\int\cos(ts)d\mu(s)\big|^2 + \big|\sum_{j=1}^{N}\sin(tx_j)\big|^2\big)\widehat{h}(t)dt.$$

In the following we only treat the cosine term, the term involving the sine can be estimated analogously. We have

$$\begin{aligned}
&\big|\partial_{x_l}\int\big(\big|\sum_{j=1}^{N}\cos(tx_j) - N\int\cos(ts)d\mu(s)\big|^2\widehat{h}(t)dt\big|^{2(\varepsilon+1)/\varepsilon}\\
&= \big|2\int\big(\sum_{j=1}^{N}\cos(tx_j) - N\int\cos(ts)d\mu(s)\big)t\sin(tx_l)\widehat{h}(t)dt\big|^{2(\varepsilon+1)/\varepsilon}\\
&\leq C\int\big|\sum_{j=1}^{N}\cos(tx_j) - N\int\cos(ts)d\mu(s)\big|^{2(\varepsilon+1)/\varepsilon}|t|^{2(\varepsilon+1)/\varepsilon}\big|\widehat{h}(t)\big|dt
\end{aligned} \tag{22}$$

where the last inequality is derived by first applying the triangle inequality and then using Jensen's inequality.
Taking now expectations, we get from Corollary 9 and the strong decay of $\widehat{h}$ that the expectation of (22) is bounded in N. This gives the claimed bound. $\square$

Proof of Theorem 3. From Propositions 13 and 15 we have that the statistics $\frac{1}{|J|}\sum_{i\in J}\mathbb{E}^h_{N,Q,\beta}G_{i,m}$ and $\frac{1}{|J|}\sum_{i\in J}\mathbb{E}_{N,V,\beta}G_{i,m}$ coincide in the limit $N\to\infty$, as long as $\lim_{N\to\infty}\frac{N^{\varepsilon_1}}{|J|}=0$ for some $\varepsilon_1>0$. It is a standard argument ([ESYY12, Section 7]) to infer from this that also the correlation measures of $P^h_{N,Q,\beta}$ and $P_{N,V,\beta}$ coincide in the sense of Theorem 3. The universality of these correlation measures is the universality result [BEY11, Corollary 2.2] which precisely states that the correlation measures of $P_{N,Q_1,\beta}$ and $P_{N,Q_2,\beta}$ have the same limit (in the sense of Theorem 3) for any real analytic and strongly convex Q_1,Q_2 with $\alpha_{Q_1},\alpha_{Q_2}>0$. □

Appendix A. Equilibrium Measures with External Fields

In this appendix, we recall some results about equilibrium measures, mainly from the book by Saff and Totik [ST97, Section I.1]. The following can be found in [ST97, Section I.1]. Let $\mathcal{M}^1(\Sigma)$ denote the set of Borel probability measures on a set Σ. Define for $\Sigma\subset\mathbb{C}$ compact the *logarithmic energy of* $\mu\in\mathcal{M}^1(\Sigma)$ as

$$I(\mu):=\int\int\log|z-t|^{-1}\,d\mu(z)d\mu(t) \tag{23}$$

and the *energy* V *of* Σ by $V:=\inf_{\mu\in\mathcal{M}^1(\Sigma)}I(\mu)$. It turns out that V is finite or ∞ and in the finite case there is a unique measure ω_Σ which minimizes (23). This measure ω_Σ is called *equilibrium measure of* Σ and the quantity $\operatorname{cap}(\Sigma):=e^{-V}$ is called *capacity of* Σ. For an arbitrary Borel set Σ we define the capacity of Σ as

$$\operatorname{cap}(\Sigma):=\sup\{\operatorname{cap}(K)\,:\,K\subset\Sigma\text{ compact}\}.$$

Lemma A.1. *If* $\Sigma=[-l,l]$, $l\geq 0$, *then* $\operatorname{cap}(\Sigma)=l/2$ *and the equilibrium measure is the arcsine distribution with support* $[-l,l]$*:*

$$d\omega_\Sigma(t)=\frac{1}{\pi\sqrt{l^2-t^2}}dt,\quad t\in[-l,l].$$

ω_Σ *has mean* 0 *and variance* $l^2/2$.

Proof. See [ST97, Section I.1]. □

Definition A.2. Let $\Sigma\subset\mathbb{R}$ be closed. Let $Q:\Sigma\longrightarrow[0,\infty]$ satisfy

a) Q is lower semicontinuous,
b) $\Sigma_0:=\{t\in\Sigma\;:\;Q(t)<\infty\}$ has positive capacity,
c) if Σ is unbounded, then $\lim\limits_{|t|\to\infty,t\in\Sigma}Q(t)-\log|t|=\infty$.

If Q satisfies these properties, we call it *external field* on Σ and $W=e^{-Q}$ its corresponding *weight function*.

Furthermore, define for $\mu\in\mathcal{M}^1(\Sigma)$ the *energy functional*

$$I_Q(\mu):=\int Q(t)d\mu(t)+\int\int\log|s-t|^{-1}\,d\mu(s)d\mu(t). \tag{24}$$

Remark A.3. In [ST97] the authors define the energy functional to be (in our notation) I_{2Q} instead of I_Q. It is more convenient for our purposes to use this definition. We note that under this change qualitative results from [ST97] remain the same but quantitative results involving Q have to be changed by a factor 2 or 1/2, respectively.

$I_Q(\mu)$ might be ∞, but the following theorem holds. The support of a measure μ will be denoted as $\operatorname{supp}(\mu)$.

Theorem A.4. *Let Q be an external field on Σ.*

a) *There is a unique probability measure $\mu_Q \in \mathcal{M}^1(\Sigma)$ with*

$$I_Q(\mu_Q) = \inf_{\mu \in \mathcal{M}^1(\Sigma)} I_Q(\mu). \tag{25}$$

b) *μ_Q has a compact support.*

c) *Let $\widetilde{Q}$ be an external field on Σ such that $\widetilde{Q} = Q$ on a compact set K with $\operatorname{supp}(\mu_Q) \subset K$ and $\widetilde{Q}(t) = \infty$ for $t \notin K$. Then $\mu_{\widetilde{Q}} = \mu_Q$.*

Proof. Statements 1) and 2) can be found in [ST97, Theorem I.1.3], 3) follows from [ST97, Theorem I.3.3] (also see remark on page 48 in [ST97]). □

μ_Q is called the *equilibrium measure* for Q. The next theorem summarizes properties of the *logarithmic potential*

$$U^\mu(z) := \int \log |z-t|^{-1} \, d\mu(t).$$

Theorem A.5.

a) *Let Q and $\widetilde{Q}$ be external fields on Σ such that $\left|Q - \widetilde{Q}\right| \leq \varepsilon$ on Σ. Then for all $z \in \mathbb{C}$*

$$\left|U^{\mu_Q}(z) - U^{\mu_{\widetilde{Q}}}(z)\right| \leq 2\varepsilon.$$

b) *Let $K \subset \mathbb{R}$ be compact and $(\mu_n)_n$ be a sequence in $\mathcal{M}^1(K)$ converging weakly to a probability measure μ. Then for a.e. $z \in \mathbb{C}$ (w.r.t. the Lebesgue measure on $\mathbb{C}$)*

$$\liminf_{n\to\infty} U^{\mu_n}(z) = U^\mu(z).$$

c) *If μ and ν are two compactly supported probability measures and their logarithmic potentials U^μ and U^ν coincide almost everywhere on $\mathbb{C}$, then $\mu = \nu$.*

Proof. Statement 1. is contained in [ST97, Corollary I.4.2], statement 2. is [ST97, Theorem I.6.9] and assertion 3. [ST97, Corollary II.2.2]. □

We also need a characterization of the support of the equilibrium measure.

Theorem A.6. *Let Q be an external field on Σ.*

a) *For a compact set K of positive capacity define the functional*

$$F_Q(K) := \log \operatorname{cap}(K) - 2 \int Q d\omega_K.$$

For any compact K of positive capacity we have $F_Q(K) \leq F_Q(\operatorname{supp}(\mu_Q))$. Furthermore, if K is compact and of positive capacity and such that $F_Q(K) = F_Q(\operatorname{supp}(\mu_Q))$, then $\operatorname{supp}(\mu_Q) \subset K$.

b) *If Q is convex, then $\operatorname{supp}(\mu_Q)$ is an interval.*

c) *If Q is even, then $\operatorname{supp}(\mu_Q)$ is even.*

Proof. For statement 1. see [ST97, Theorem IV.1.5], for statements 2. and 3. [ST97, Theorem IV.1.10]. □

Theorem A.7.

a) *Let Q be an external field on Σ. If Q is finite on* $\operatorname{supp}(\mu_Q)$ *and locally of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$ (which means that Q is continuously differentiable and the derivative Q' is Hölder continuous with parameter ε), then μ_Q a continuous density on the interior of* $\operatorname{supp}(\mu_Q)$.

b) *If Q has two Lipschitz derivatives and is strictly convex, then* $\operatorname{supp}(\mu_Q) =: [a, b]$ *and the density of μ_Q can be represented as*

$$\frac{d\mu(t)}{dt} = r(t)\sqrt{(t-a)(b-t)}\mathbb{1}_{[a,b]}(t), \tag{26}$$

where r can be extended into an analytic function on a domain containing $[a, b]$ and $r(t) > 0$ for $t \in [a, b]$. In particular, density is positive on (a, b).

Proof. Statement 1. is [ST97, Theorem IV.2.5], for assertion 2. see e.g. the appendix of the paper by McLaughlin and Miller [MM08]. □

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